SOLUTION SHEET 9:

1. $\Leftarrow$: $K(\alpha) = SF_K(x^n - a)$ $\quad x^n - a = x^n - b^r c^n = x^n - (\beta^r \cdot c)^n$
$\Rightarrow \beta^r \cdot c \in SF_K(x^n - a) \Rightarrow \beta^r \in SF_K(x^n - a)$ as $(r,n) = 1$
there exist k st $r \cdot k = 1 + q \cdot n$ for some $q \Rightarrow$
$(\beta^r)^k = \beta . b^q \in K(\alpha) \Rightarrow \beta \in K(\alpha) \Rightarrow [K(\alpha) : K(\beta)] = 1$

$\Rightarrow$: Note that as $K(\beta)/K$ is Kummer, $|Kum(K(\beta)/K)| = |Gal(K(\beta)/K)|$ & $|Gal(K(\beta)/K)| = [K(\beta) : K] = n$ therefore $|Kum(K(\beta)/K)| = n$.
Now $1, \beta, \beta^2, -, \beta^{n-1}$ give n distinct classes in $Kum(K(\beta)/K)$, however α also gives a non-trivial class in $Kum(K(\beta)/K)$ therefore $[\alpha] = [\beta^k]$ in $Kum(K(\beta)/K)$. Recall that there is an injective morphism

$$\begin{aligned} Kum(K(\beta)/K) &\xrightarrow{f} K^\times/(K^\times)^n \\ [x] &\rightarrow x^n . (K^\times)^n \end{aligned}$$

therefore $f([\alpha]) = f([\beta^k]) \Rightarrow \alpha^n = (\beta^k . c)^n$ for some $c \in K^\times$
moreover $(k,n) = 1$ as otherwise $K(\alpha)/K$ has degree less than n.

2. It is clear that the extension is 4-Kummer.
By the course we know that $[L:K] = |Kum(L/K)|$. Moreover, there is an injective group morphism

$$\begin{aligned} Kum(L/K) &\xhookrightarrow{f} K^\times/(K^\times)^4 \\ aK^\times &\longmapsto a^4 (K^\times)^4 \end{aligned}$$

Therefore we count the number of elements in $f(Kum(L/K))$. Notice that
$f(Kum(L/K)) = \{ (xyz)^i (yz)^j . (xz^2)^l \mid i,j,l \in \{0,1,2,3\}\} / (K^\times)^4$.
that is it is subgroup of $(K^\times)/(K^\times)^4$ generated by xyz, yz, xz^2.
Observe that the subgroup generated by xyz and yz has order 16.
Now

$$(xyz)^2 . (yz)^2 . (xz^2)^2 = x^4 y^8 z^8 \in (K^\times)^4$$

Therefore either the subgroup generated by xyz, yz is equal to the one generated by xyz, yz, xz^2 or it has index 2. One can see that these groups are not equal by showing that xz^2 is not contained in the subgroup generated by xyz, yz therefore $|f(Kum(L/K))| = 32 = [L:K]$.

3. As before this is a Kummer extension and the degree of the extension is given by the number of elements

$$\{ p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} \mid a_i \in \{0,1\}\} / (\mathbb{Q}^\times)^2$$

but $p_1^{a_1} \ldots p_n^{a_n}$ with $a_i \in \{0,1\}$ is never a square so the number of elements in this set is 2^n.

4. Let $\tau \in Aut(\bar{\mathbb{Q}})$ be a non-trivial torsion element and let $G = \langle \tau \rangle$ be the subgroup of $Aut(\bar{\mathbb{Q}})$ generated by τ. Then $[\bar{\mathbb{Q}} : \bar{\mathbb{Q}}^G] < \infty$ in fact $[\bar{\mathbb{Q}} : \bar{\mathbb{Q}}^G] = |G|$. By Artin-Schreier $\bar{\mathbb{Q}} = \bar{\mathbb{Q}}^G(i)$ where $i^2 = -1$. Therefore $[\bar{\mathbb{Q}} : \bar{\mathbb{Q}}^G] = [\bar{\mathbb{Q}}^G(i) : \bar{\mathbb{Q}}^G] = 2 \Rightarrow |G| = 2$
$\Rightarrow \tau$ has order 2.